CALCULATION OF PLASTIC STRIPS AT THE CORNER POINTS

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Abstract

In this article represented are the calculations of initial plastic strips-zones, developed from the corner points of elastoplastic body, which are stress concentrators. With the aim of conducting of these calculations there has been made a transition from the problem on plastic zones to the problems of the theory of elasticity for the wedge-shaped body with the rupture displacement line at the vertex. The exact solutions of the above-mentioned problems are constructed by Wiener-Hopf method. Based on these solutions the lengths of plastic strips and the directions of their development are defined. The results of symmetric cases of the crack end, which is on the media-separating boundary, and the corner point of the hole brings us to a conclusion that the process of development of the initial plastic zone has two stages. At the first stage two side plastic strips are developed from the corner point, and at the second stage one more additional strip, which is considerably smaller than the side ones. In the case of a hole at the second stage before the appearance of the third strip, the side plastic strips deviate from their primary direction of development, thus, leading to the angle increase in between them.

Introduction

An approach to the investigation of corner points of elastoplastic body from the point of view of initial development of plastic zones near them under the conditions of plane problem is proposed. Following the widely-used and confirmed by numerous experiments localization hypothesis, the initial plastic zones are modeled by narrow rectilinear plastic strips, emerging from corner points. These strips are plastic slip lines or plastic Dugdale’s lines.

The essence of the approach proposed is in the reduction of the mentioned question, concerning to the corner point under investigation, to the static problem of the theory of elasticity for wedge-shaped region with rectilinear cut of finite length, emerging from corner point, and nonclassical condition at infinity, which allows to take into account the influence of external field; in construction by Wiener-Hopf method an exact solution of this problem and in determination on its base the length and the direction of initial development of the plastic strip.

The approach proposed is used for calculation of initial plastic zones near the corner point of the hole, of the rigid inclusion, of the media-separating boundary, of the intersection of slip lines under the conditions of symmetrical problem in limits of the model with two slip lines; near the end of the crack in limits of “trident” model; near the corner point of a rigid punch, impressed into an elastoplastic body, in limits of a model with only slip line; near the end of the crack at the media-separating boundary in limits of a model with only slip line and a model with only Dugdale’s line. İn perspective using the given approach, whole classes of new problems of initial plastic strips, emerging from corner points, can be investigated.

Here are some examples of problem solution of the above-mentioned type.

Under calculations of the initial plastic zones at the ends of cracks and other corner points – concentrators of stresses in the elastoplastic body under the conditions of plane strain in the limits of models with the rupture displacement lines the most spread model is with two slip lines [1-4]. The only admitted is the rupture of tangential displacement, and the tangential stress equals to shear yield point. However, the results of some theoretical and experimental research work, conducted recently, prove the evidence of the fact that the process of the initial development of the plastic zone near the corner point – the concentrator of stresses in many cases the “trident” model describes more exactly [5,6]. According to the model, emerging are the two slip lines from the corner point and one more rupture displacement line of the considerably shorter length.

Below the plane symmetric problems on the calculations within the limits of the “trident” model of the plastic zone at the end of the crack, that is on the interface of two different homogeneous isotropic elastoplastic media with Young's modulus $E_{1,2}$ and Poisson's ratio $\nu_{1,2}$, under the condition that the binding material is more plastic than the materials of contacting bodies, and also at the corner point of the hole in homogeneous isotropic elastoplastic body are considered.

Formulation of the problems

Plastic strips at the end of the crack that is on the media-separating boundary

In this case we come down to a plane static symmetrical problem of the theory of elasticity for the piece-homogeneous isotropic plane with the media-separating boundary in the angle 2α form, from the corner point of which emerge semi-infinite crack, two slip lines which are on the interface and line D of the rupture of displacement of considerably shorter length that is inside of the angle (Figure1).

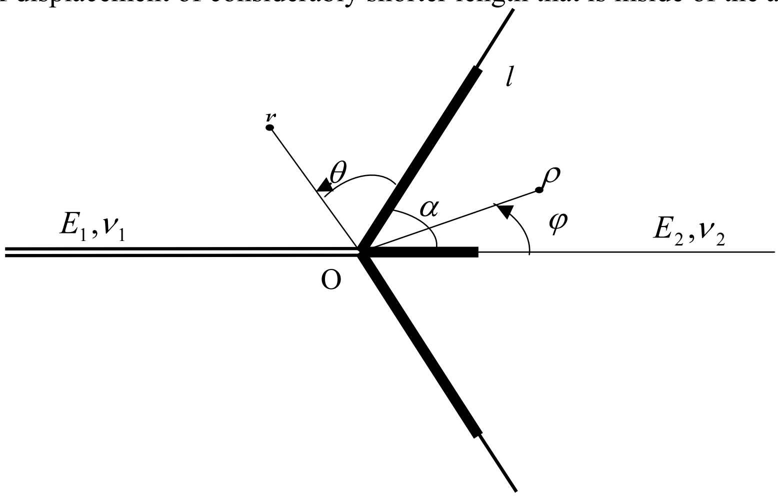

Fig. 1. Rupture displacement lines at the end of the crack in the piece-homogeneous plane

On line D the normal stress equals to the given constant of the material σ. At infinity the asymptotic is realized, represented by itself the solution of the analogous problem without slip lines and line D, which corresponds to the smallest in the interval]-1;0[root $\lambda_0(\alpha, \nu_1, \nu_2, E_1/E_2)$ of its characteristically equation. This solution contains arbitrary constant C, that is considered to be given. Constant C characterizes the intensity of the external field and must be determined from the solution of the external problem.

Since the length d of line D is considerably shorter than the length l of slip lines, the formulated problem of the theory of elasticity (the problem in the whole) can be divided into external and internal problems. The external problem is the problem analogous to the problem in the whole without line D. The internal problem is the problem analogous to the problem in the whole with semi-infinite slip lines. At infinity in the internal problem the principal terms of the expansion of stresses in asymptotic series coincide with principal terms of the expansion of stresses in asymptotic series in the external problem near the end of the crack.

The boundary conditions of the external problem are as follows:

$$\theta = \pi - \alpha,\ \sigma_\theta = \tau_{r\theta} = 0; \quad \theta = -\alpha,\ \tau_{r\theta} = 0,\ u_\theta = 0$$

$$\theta = 0,\ \langle \sigma_\theta \rangle = \langle \tau_{r\theta} \rangle = 0,\ \langle u_\theta \rangle = 0 \tag{1}$$

$$\theta = 0,\ r < l,\ \tau_{r\theta} = \tau_s; \quad \theta = 0,\ r > l,\ \langle u_r \rangle = 0 \tag{2}$$

($-\alpha \le \theta \le \pi - \alpha$; $\langle a \rangle$ is the jump of value a; τ_s is shear yield point).

The solution of the formulated problem with boundary conditions (1), (2) are the sum of solutions of the two problems. The first differs, that instead of the first condition (2) we have

$$\theta = 0,\ r < l,\ \tau_{r\theta} = \tau_s - Cgr^{\lambda_0} \tag{3}$$

($g(\alpha, \nu_1, \nu_2, E_1/E_2)$ is the known function), whereas at infinity the stresses decrease as $o(1/r)$.

The second problem is the analogous problem without slip lines, the solution of that is known.

Using Mellin's integral transform and taking into account the second condition (2) and condition (3), the first problem is reduced to the following Wiener-Hopf functional equation [7, 8]:

$$\Phi^+(p) + \frac{\tau_s}{p+1} + \frac{\tau}{p+\lambda_0+1} = -A\ tg(p\pi)\ G(p)\Phi^-(p) \qquad (-\varepsilon_1 < \mathrm{Re}\ p < \varepsilon_2) \tag{4}$$

$(-\varepsilon_1 < \mathrm{Re}\, p < \varepsilon_2)$

$$A = \frac{(1+æ_1)[1+æ_1+(1+æ_2)e]}{2[æ_1+(1+æ_1æ_2)e+æ_2e^2]}, \quad G(p) = \frac{\Delta_1(p)\cos p\pi}{\Delta_2(p)\sin p\pi}$$

$$\Delta_1(p) = 2[æ_1+(1+æ_1æ_2)e+æ_2e^2][a_0(p)+a_1(p)e], \quad \Delta_2(p) = [1+æ_1+(1+æ_2)e][b_0(p)+b_1(p)e+b_2(p)e^2]$$

$$a_0(p) = (1+æ_1)(\sin 2p\alpha + p\sin 2\alpha)[\sin 2p(\pi-\alpha) - p\sin 2\alpha]$$

$$a_1(p) = 2(1+æ_2)(\cos 2p\alpha - \cos 2\alpha)[\sin^2 p(\pi-\alpha) - p^2\sin^2\alpha]$$

$$b_0(p) = (\sin 2p\alpha + p\sin 2\alpha)\left\{(1+æ_1)^2 - 4[æ_1\sin^2 p(\pi-\alpha) + p^2\sin^2\alpha]\right\}$$

$$b_1(p) = (1+æ_1)(1+æ_2)\sin 2p\pi + 4(æ_2\sin 2p\alpha - p\sin 2\alpha)[\sin^2 p(\pi-\alpha) - p^2\sin^2\alpha] -$$

$$-(\sin 2p\alpha + p\sin 2\alpha)\left\{(1+æ_1)(1+æ_2) - 4[æ_1\sin^2 p(\pi-\alpha) + p^2\sin^2\alpha]\right\}$$

$$b_2(p) = -4(æ_2\sin 2p\alpha - p\sin 2\alpha)[\sin^2 p(\pi-\alpha) - p^2\sin^2\alpha]$$

$$æ_{1,2} = 3-4\nu_{1,2}, \quad e = \frac{1+\nu_2}{1+\nu_1}\frac{E_1}{E_2}, \quad \tau = -Cgl^{\lambda_0}$$

$$\Phi^+(p) = \int_1^\infty \tau_{r\theta}(xl,0)x^p\,dx, \quad \Phi^-(p) = \frac{E_1}{4(1-\nu_1^2)}\int_0^1 \left\langle\frac{\partial u_r}{\partial r}\right\rangle\Bigg|_{\substack{r=xl\\ \theta=0}} x^p\,dx$$

($\varepsilon_{1,2}$ are sufficiently small positive numbers).

The solution of the equation (4) is as follows [8]

$$\Phi^+(p) = -\frac{pG^+(p)}{K^+(p)}\left\{\frac{\tau_s}{p+1}\left[\frac{K^+(p)}{pG^+(p)} + \frac{K^+(-1)}{G^+(-1)}\right] + \frac{\tau}{p+\lambda_0+1}\times\left[\frac{K^+(p)}{pG^+(p)} + \frac{K^+(-\lambda_0-1)}{(\lambda_0+1)G^+(-\lambda_0-1)}\right]\right\} \quad (\mathrm{Re}p<0) \quad (5)$$

$$\Phi^-(p) = \frac{K^-(p)G^-(p)}{A}\left[\frac{\tau_s K^+(-1)}{(p+1)G^+(-1)} + \frac{\tau K^+(-\lambda_0-1)}{(p+\lambda_0+1)(\lambda_0+1)G^+(-\lambda_0-1)}\right] \quad (\mathrm{Re}\, p < 0)$$

$$\exp\left[\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\ln G(z)}{z-p}dz\right] = \begin{cases} G^+(p), & \mathrm{Re}\, p<0 \\ G^-(p), & \mathrm{Re}\, p>0 \end{cases}, \quad K^{\pm}(p) = \frac{\Gamma(1\mp p)}{\Gamma(1/2 \mp p)}$$

($\Gamma(z)$ is gamma function). From (5) we get the asymptotic

$$\Phi^-(p) \sim \frac{1}{A\sqrt{p}}\left[\frac{\tau_s K^+(-1)}{G^+(-1)} + \frac{\tau K^+(-\lambda_0-1)}{(\lambda_0+1)G^+(-\lambda_0-1)}\right] \quad (p\to\infty) \quad (6)$$

According to [1]

$$\theta=0,\ r\to l+0,\ \tau_{r\theta} \sim \frac{æ_1+e+1+æ_2e}{2(1+æ_2e)}\frac{k_{II}}{\sqrt{2\pi(r-l)}}; \quad \theta=0,\ r\to l-0,\ \left\langle\frac{\partial u_r}{\partial r}\right\rangle \sim -\frac{4(1-\nu_1^2)}{E_1}\frac{æ_1+e}{1+æ_1}\frac{k_{II}}{\sqrt{2\pi(l-r)}}$$

(k_{II} is the stress intensity factor at the end of slip line). Applying Abel's type theorem [7] we find the asymptotics

$$\Phi^+(p) \sim \frac{æ_1+e+1+æ_2e}{2(1+æ_2e)}\frac{k_{II}}{\sqrt{-2pl}}, \quad \Phi^-(p) \sim -\frac{æ_1+e}{1+æ_1}\frac{k_{II}}{\sqrt{2pl}} \quad (p\to\infty) \quad (7)$$

According to (6), (7)

$$k_{II} = \frac{2\sqrt{2}(1+æ_2e)}{æ_1+e+1+æ_2e}\left[\frac{g\Gamma(\lambda_0+1)}{\Gamma(\lambda_0+3/2)G^+(-\lambda_0-1)}Cl^{\lambda_0+1/2} - \frac{2}{\sqrt{\pi}G^+(-1)}\tau_s\sqrt{l}\right] \quad (8)$$

The length of the slip line is determined from the condition of continuity of stresses at its end. Equating to zero the right side of (8), we come to the equation for determination of slip line length.

The internal problem is reduced to the following Wiener-Hopf functional equation:

$$\Phi^+(p) + \frac{\sigma}{p+1} + \frac{\sigma_1}{p+\lambda_1+1} = -tg(p\pi)G(p)\Phi^-(p) \quad (-\varepsilon_1 < \mathrm{Re}\, p < \varepsilon_2) \quad (9)$$

$$G(p) = \frac{\Delta_3(p)\cos p\pi}{\Delta_4(p)\sin p\pi}$$

$$\Delta_3(p)=2(1+æ_1)[\sin 2p(\pi-\alpha)-p\sin 2\alpha](\sin^2 p\alpha-p^2\sin^2 +2(1+æ_2)e(\sin 2p\alpha+p\sin 2\alpha)[\sin^2 p(\pi-\alpha)-p^2\sin^2\alpha]$$

$$\Delta_4(p)=(1+æ_1)(\sin 2p\alpha+p\sin 2\alpha)[\sin 2p(\pi-\alpha)-p\sin 2\alpha]+2(1+æ_2)e(\cos 2p\alpha-\cos 2\alpha)[\sin^2 p(\pi-\alpha)-p^2\sin^2\alpha]$$

$$\sigma_1=F\tau_s\left(\frac{C}{\tau_s}\right)^{\lambda_1/\lambda_0}d^{\lambda_1}$$

$$\Phi^+(p)=\int_1^\infty \sigma_\varphi(xd,0)x^p\,dx\,,\quad \Phi^-(p)=\frac{E_2}{2(1-\nu_2{}^2)}\int_0^1 \left.\frac{\partial u_\varphi}{\partial\rho}\right|_{\substack{\rho=xd\\ \varphi=0}} x^p\,dx$$

($\lambda_1(\alpha,\nu_1,\nu_2,E_1/E_2)\in]-1;0[$ is the exponent of singularity of stresses at the corner point of the external problem; $\mathrm{F}(\alpha,\nu_1,\nu_2,E_1/E_2)$ is the known function).

Based on the solution of the equation (9) the line D length is defined.

Plastic strips at the corner point of the hole

In this case we come down to the plane static symmetrical problem of the theory of elasticity for homogeneous isotropic wedge with stress-free sides and the angle bigger than π, from the vertex of which emerge two straight slip lines and the line of the rupture of normal displacement is considerably shorter in length (Figure2).

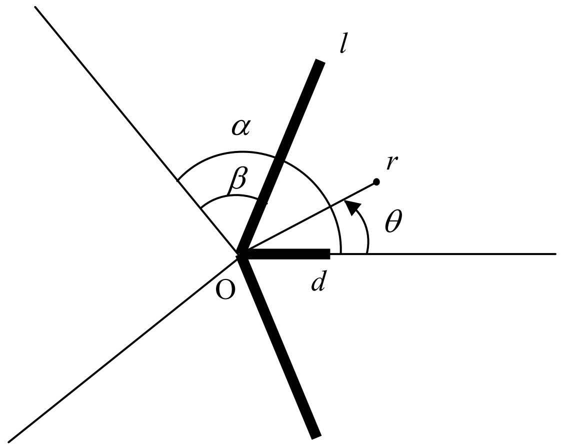

Fig. 2. Rupture displacement lines at the wedge vertex

The boundary conditions of the problem are as follows:

$\theta=0,\ r<d,\ \sigma_\theta=\sigma,\ \tau_{r\theta}=0;\ \theta=0,\ r>d,\ \tau_{r\theta}=0,\ u_\theta=0$

$\theta=\alpha-\beta,\ r<l,\ \langle\sigma_\theta\rangle=\langle\tau_{r\theta}\rangle=0,\ \langle u_\theta\rangle=0,\ \tau_{r\theta}=\tau_s$

$\theta=\alpha-\beta,\ r>l,\ \langle\sigma_\theta\rangle=\langle\tau_{r\theta}\rangle=0,\ \langle u_\theta\rangle=\langle u_r\rangle=0$

$\theta=\alpha,\ \sigma_\theta=\tau_{r\theta}=0\qquad(0\le\theta\le\alpha)$

The external and the internal problems are reduced to Wiener-Hopf equations with the coefficients

$$G_e(p)=\frac{\delta(p)\cos p\pi}{(\sin 2p\alpha+p\sin 2\alpha)\sin p\pi}\ ,\quad G_i(p)=\frac{\Delta(p)\cos p\pi}{\delta(p)\sin p\pi}$$

$$\delta(p)=(\sin 2p\beta+p\sin 2\beta)[\sin 2p(\alpha-\beta)+p\sin 2(\alpha-\beta)+2[\cos 2p(\alpha-\beta)-\cos 2(\alpha-\beta)](\sin^2 p\beta-p^2\sin^2\beta)$$

$$\Delta(p)=2\{(\sin 2p\beta+p\sin 2\beta)[\sin^2 p(\alpha-\beta)-p^2\sin^2(\alpha-\beta)]+[\sin 2p(\alpha-\beta)+p\sin 2(\alpha-\beta)](\sin^2 p\beta-p^2\sin^2\beta)\}$$

Based on the solutions of the given equations the lengths of rupture displacement lines and the directions of their development are determined. At this, the direction of slip line development is defined on the condition of maximum of sum of lengths of rupture displacement lines.

Results

These are the following formulas that serve to define the lengths of plastic strips emerging from the end of the crack, reaching the media-separating boundary:

$$l=F_1\left(\frac{C}{\tau_s}\right)^{-1/\lambda_0}\ ,\quad d=F_2\left(\frac{C}{\tau_s}\right)^{-1/\lambda_0}\left(\frac{\tau_s}{\sigma}\right)^{-1/\lambda_1}\qquad(10)$$

Some values of $\lambda_0,\lambda_1,F_1,F_2$ are given in the Table 1.

Table 1. Some values of functions in the formulas for the lengths of plastic strips at the end of the crack in the piece-homogeneous body

E_1/E_2	ν_1	ν_2		α^o							
				10	30	50	70	90	110	130	150
0,5	0,25	0,25	λ_0	-0.488	-0,464	-0,446	-0,432	-0,430	-0,448	-0,476	-0,495
			λ_1	-0.480	-0.438	-0.375	-0.260	0	-0.306	-0.441	-0.489
			F_1	0.072	0.739	3.365	6.176	3.507	0.525	0.023	$1.56\cdot10^{-4}$
			F_2	134.14	313.37	890.29	9602.1		219.64	3.71	0.12
	0,33	0,33	λ_0	-0.491	-0.474	-0.458	-0.443	-0.437	-0.450	-0.477	-0.495
			λ_1	-0.480	-0.438	-0.375	-0.260	0	-0.306	-0.441	-0.489
			F_1	0.144	1.2782	4.096	6.034	3.158	0.457	0.019	$1.27\cdot10^{-4}$
			F_2	109.02	255.4	712.16	7700.8		184.86	3.05	0.097
2	0,25	0,25	λ_0	-0.515	-0,545	-0,571	-0,583	-0,576	-0,554	-0,529	-0,509
			λ_1	-0.434	-0.351	-0.283	-0.197	0	-0.313	-0.451	-0.493
			F_1	22.782	77.653	59.236	25.802	7.478	1.295	0.099	$1.33\cdot10^{-4}$
			F_2	5136.8	14587	32404	$6.08\cdot10^{5}$		143.16	3.13	0.12
	0,33	0,33	λ_0	-0.512	-0.539	-0.565	-0.579	-0.574	-0.553	-0.529	-0.509
			λ_1	-0.434	-0.351	-0.283	-0.197	0	-0.313	-0.451	-0.493
			F_1	16.424	59.449	46.31	19.89	5.631	0.961	0.074	$1.03\cdot10^{-3}$
			F_2	4269.3	13045.9	29073.2	$5.23\cdot10^{5}$		106.33	2.35	0.093

Formulas (10) set up the law of development of the initial plastic zone near the considerate corner point.

Established is the following two stage mechanism of the process of development of the initial plastic zone near the corner point of the hole. At the first stage appear two plastic strips emerging from the corner point and forming the angle with body boundary which increases with the angle increasein between the boundary lines. The strips grow with the loading increase. At the second stage the side plastic strips deviate from the initial direction of development, which is leading to the increase of the angle in between them, and the third strip appears, developing from the corner point, which is considerably smaller than the side ones. The angle between the side plastic strip and the body boundary increases with the increase of the angle between the boundary lines. The bigger is σ, the bigger is the length of the side strips and the smaller is the length of the third strip, and the angle perturbation between the side strip and the boundary, created by the third strip. With the loading increase, all the three strips grow. The initial stage of the development process of the plastic zone at the corner point ends when the length of the side plastic strips strops being of considerably smaller than the body sizes.

With the above-described method could be calculations done within the limits of the "trident" model of the initial plastic zones near different types of other corner points of sophisticated nature in elastoplastic body.

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